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RADIAL VARIATION OF FUNCTIONS IN BESOV SPACES

David Walsh

Abstract

This paper considers the radial variation function $F(r,t)$ of an analytic function $f(z)$ on the disc D. We examine $F(r,t)$ when f belongs to a Besov space A^s_{pq} and look for ways in which F imitates the behaviour of f. Regarded as a function of position (r,t) in D, we show that F obeys a certain integral growth condition which is the real variable analogue of that satisfied by f. We consider also the radial limit $F(t)$ of F as a function on the circle. Again, $F \in B^s_{pq}$ whenever $f \in A^s_{pq}$, where B^s_{pq} is the corresponding real Besov space. Some properties of F are pointed out along the way, in particular that $F(r,t)$ is real analytic in D except on a small set. The exceptional set E on the circle at which $\lim_{r\to1} f(re^{it})$ fails to exist, is also considered; it is shown to have capacity zero in the appropriate sense. Equivalent descriptions of E are also given for certain restricted values of p, q, s.

1. Introduction

In [**4**] A. Beurling considered functions f on the unit circle T which belong to a certain Besov space $B^{1/2}_2$, and described the set E of points e^{ix} for which $\lim_{r\to1} f(re^{ix})$, the radial limit of the Poisson integral of f on the unit disc D, fails to exist. He showed that the set E coincides with the set of points on T for which the Fourier series of f diverges, and that this in turn coincides with the set of points for which the symmetric derivative

$$\lim_{h\to0} \frac{1}{2h} \int_{x-h}^{x+h} f(t)\, dt,$$

fails to exist. A consequence of his approach is that E has logarithmic capacity zero. Beurling associated with an analytic function $f \in B^{1/2}_2$,

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its radial variation

$$F(r,t) = \int_0^r |f'(ue^{it})|\,du, \quad r < 1.$$

He proved that F has certain properties closely resembling those of f and made decisive use of these properties in deriving the result on capacity mentioned above. He showed that

$$\iint_D |\nabla F(r,t)|^2 r\,dt\,dr \le \iint_D |f'(re^{it})|^2 r\,dt\,dr.$$

It is clear from the definition, that the boundary function $F(t) = \lim_{r\to 1} F(r,t)$ exists, finite or infinite, for all $t \in [-\pi,\pi]$. Beurling showed further that $F \in B_2^{1/2}$, and that its norm in that space satisfies

$$||F||_B \le ||f||_B.$$

He applied these results to show that the set of points on T for which $F(t) = \infty$ has logarithmic capacity zero; it is well known that this capacity is naturally associated with the space $B_2^{1/2}$ or with the Dirichlet space $A_2^{1/2}$, the subspace of $B_2^{1/2}$ consisting of analytic functions.

Since then, these results have been extended in various directions. In [**10**] Nagel, Rudin and Shapiro considered Bessel potential spaces L_s^p. They showed that for $w \in T$, $\lim_{z\to w} f(z)$ exists a.e. for wider approach regions than the non-tangential. In fact for the case of the Dirichlet space $L^2_{1/2}$, the approach region can have exponential order of contact with T. Ahern and Cohn [**3**] considered similar spaces of holomorphic functions on the ball of $\mathbf{C_n}$ called Hardy-Sobolev spaces. Certain admissible approach regions are first defined and the exceptional set $E(f)$ is the set of points w on the boundary for which $\lim_{z\to w} f(z)$ fails to exist as $z \to w$ within this region. They showed that the exceptional set has capacity zero in the appropriate sense. Here the capacities are Bessel capacities.

Efforts have also been made to show that the estimate on the exceptional set is sharp, that is given a compact subset K of capacity zero, there is a function f in the space for which $K = E(f)$. In this regard, [**3**] showed that the sets of Bessel capacity zero completely characterize the exceptional sets for the case $n = 1$. Further results of this type were proved by Cohn and Verbitsky [**6**]. A great impetus to developments in this area was given by Carleson's book [**5**]. In particular this demonstrated how sets of Cantor type can be used to prove that certain statements about exceptional sets are sharp.

More recently, for a function f in the Dirichlet space, Twomey [**16**] has exhibited tangential approach regions A_γ such that f has A_γ-limits at all boundary points outside of a set of logarithmic capacity zero. Moreover he showed that such approach regions are in a certain sense optimal. In yet another direction the result on the exceptional set for the radial variation $F(t)$ has been extended to certain weighted Dirichlet spaces. See [**17**] and the references cited therein. Indeed all the works cited above contain other pertinent references.

Our aim is to try to extend in so far as we can the results of Beurling on the radial variation function $F(r,t)$ to a class of Besov spaces B^s_{pq}, for which $B^{1/2}_2$ is the special case $s = 1/2$, $p = 2 = q$. In Section 2 some properties of $F(r,t)$ are set forth which are needed later. It turns out that $F(r,t)$ is an analytic function in D outside of a small set H which is determined by the zeros of f' of odd order.

In Section 3 we explore the consequences for $F(r,t)$ as a function on D, of the assumption that $f \in A^s_{pq}$. For $0 < s < 1$, it is shown in Theorem 1 that F obeys a certain integral growth condition which is the real variable analogue of that satisfied by f. For $s = 1$, we obtain a limited result (Theorem 3), whereby we require $p = 1$, $1 \le q < 2$.

We consider the boundary function $F(t)$ in Section 4. For $0 < s < 1$, $1 \le p, q < \infty$, it is shown in Theorem 4 that $f \in A^s_{pq}$ implies that $F \in B^s_{pq}$; Theorem 1 is used in the proof. For $s = 1$, $1 \le q < 2$, we are able to show in a like manner, by means of Theorem 3, that $f \in A^1_{1q}$ implies that $F \in B^1_{1q}$.

In Section 5 we consider the exceptional subset E' of T, for which $F(t) = \infty$. The capacity $C(\cdot; B^s_{pq})$ is introduced, and it is shown that if $f \in A^s_{pq}$, $1 < p, q < \infty$, then $C(E'; B^s_{pq}) = 0$. An immediate application follows. Let $E(f)$ be the set of points for which the radial limit, $\lim_{r\to1} f(re^{it})$, fails to exist; then $C(E; B^s_{pq}) = 0$. It should be noted that for $s > 1/p$, B^s_{pq} is a space of continuous functions and therefore this last result is significant only for $s \le 1/p$. For the special case $s = 1/p$, $q = p$, $1 < p \le 2$, we show further that the alternative characterizations of E obtained by Beurling, also hold.

It is not to be expected that the very strong results last cited hold without restriction on s, p, q. Certainly nothing of this kind can be expected for $s < 1/p$, since the diagonal Besov spaces $B^{1/p}_p$ with $s = 1/p$, $p = q$, are well known as the interface between the smoother spaces where $s > 1/p$ and the less tractable class with $s < 1/p$, where many results break down.

1.1. Preliminaries.

Let D denote the unit disc, T the unit circle in the complex plane. For convenience we shall let m denote normalised Lebesgue measure on the circle T, and m_2 the normalised area measure on the disc. Given a function f on T, let $\Delta_t f(e^{ix}) = f(e^{i(x+t)}) - f(e^{ix})$, and $\Delta_t^k = \Delta_t(\Delta_t^{k-1})$, denote the differences of order one and order k respectively, of f at e^{ix}. Let

$$f(re^{it}) = \int_{-\pi}^{\pi} f(e^{ix}) P_r(x-t)\, dm(x)$$

be the Poisson integral of f on the disc, where $P_r(t)$ is the Poisson kernel. Suppose now that f is analytic in D. If $0 \le r < 1$, let

$$M_p(f,r) = \left(\int_{-\pi}^{\pi} |f(re^{it})|^p \, dm \right)^{1/p}, \quad (0 < p < \infty),$$

denote the integral mean of f of order p. It is well known that $M_p(f,r)$ is an increasing function of r on $[0,1)$ and that the class of functions f for which $\sup_{r<1} M_p(f,r) < \infty$, is the familiar Hardy space H^p [**7**]. Given $f(e^{it}) \sim \sum_{-\infty}^{\infty} a_n e^{int}$, we write

$$s_n(f)(e^{it}) = \sum_{k=-n}^{n} a_k e^{ikt} \quad (n \ge 0),$$

for the partial sums of the Fourier series of f. For $1 \le p, q < \infty$, $s > 0$, and an arbitrary integer $m > s$, we define the Besov space B_{pq}^s by

$$B_{pq}^s = \left\{ f \in L^p : \int_{-\pi}^{\pi} \frac{||\Delta_t^m f||_p^q}{|t|^{1+sq}} \, dm(t) < \infty \right\}, \quad 1 \le q < \infty.$$

It is well known that the definition is independent of m. For a discussion of these spaces see [**1**], [**11**], [**14**], [**15**]. It is known that the Riesz projection is a bounded operator from B_{pq}^s to itself. Let A_{pq}^s denote the subspace of B_{pq}^s consisting of analytic functions. The space A_{pq}^s may be characterized as follows: the analytic function $f \in A_{pq}^s$ if and only if

$$||f||_A = ||f||_p + \left\{ \int_0^1 (1-r^2)^{q(m-s)-1} M_p(f^{(m)}, r)^q r \, dr \right\}^{1/q} < \infty.$$

Once again the definition is independent of m for $m > s$. Each function $f \in A_{pq}^s$ is in H^p and has a boundary function, also denoted by f, on T. This boundary function is in B_{pq}^s and we denote its norm in that space by $\|f\|_B$. Of course the two norms are equivalent.

2. Properties of F

2.1. $F(r,t)$ a real analytic function.

Recall that for f analytic,

$$F(r,t) = \int_0^r |f'(ue^{it})|\,du, \quad r < 1. \tag{1}$$

Since $f(re^{it}) - f(0) = \int_0^r f'(ue^{it})\,du$, it is clear that

$$|f(re^{it})| \le |f(0)| + F(r,t), \quad r < 1, \quad -\pi \le t \le \pi,$$

and $F(r,t)$ is a majorant for f. The function $F(r,t)$ represents the length of the image of the radius vector $[0, re^{it}]$ under the mapping f, and is known as the radial variation. An immediate property of F is that if $F(t) = \lim_{r\to 1} F(r,t) < \infty$, then $\lim_{r\to 1} f(re^{it})$ exists.

The gradient of F is $\nabla F(r,t) = \left(\frac{\partial F}{\partial r}, 1/r\frac{\partial F}{\partial t}\right) = \left(|f'(re^{it})|, 1/r\frac{\partial F}{\partial t}\right)$. We shall make frequent use of two elementary facts:

Lemma 1. *Let $G(r,t)$ be a differentiable function of r and suppose that $\frac{\partial |G|}{\partial r}(r,t)$ exists. Then*

$$\left|\frac{\partial |G|}{\partial r}(r,t)\right| \le \left|\frac{\partial G}{\partial r}(r,t)\right|.$$

Proof: For,

$$\left|\frac{|G(r+h,t)| - |G(r,t)|}{h}\right| \le \left|\frac{G(r+h,t) - G(r,t)}{h}\right|,$$

for any $h \ne 0$. Taking limits as $h \to 0$, the result follows. □

Lemma 2. *Suppose that f is an analytic function in the disc, $0 < r < 1$, and $\frac{\partial |f|}{\partial t}(re^{it})$ exists. Then*

$$\left|\frac{\partial |f|}{\partial t}(re^{it})\right| \le r|f'(re^{it})|.$$

We look next at some partial derivatives of F. It is obvious that $\frac{\partial F}{\partial r} = |f'(re^{it})|$ for all points (r,t) in the disc. Although $\frac{\partial |f'|}{\partial t}(re^{it})$ does not exist at points where $f'(z)$ has a zero of odd order, these points are at most countably infinite and the following result still holds:

Lemma 3. *For all points (r,t) in the disc,*

$$\frac{\partial F}{\partial t}(r,t) = \int_0^r \frac{\partial |f'|}{\partial t}(ue^{it})\,du.$$

Proof: Consider the difference quotient

$$\frac{F(r,t+h)-F(r,t)}{h}=\int_0^r \frac{|f'(ue^{i(t+h)})|-|f'(ue^{it})|}{h}\,du,$$

whence

$$\left|\frac{F(r,t+h)-F(r,t)}{h}\right| \le \int_0^r \left|\Big(|f'(ue^{i(t+h)})|-|f'(ue^{it})|\Big)1/h\right| du$$
$$\le \int_0^r \left|\Big(f'(ue^{i(t+h)})-f'(ue^{it})\Big)1/h\right| du.$$

By the Mean Value Theorem, $f'(ue^{i(t+h)})-f'(ue^{it}) = h\frac{\partial f'}{\partial s}(ue^{is}) = f''(ue^{is})iue^{is}h$, for some s between t and $t+h$. Since $|f''(ue^{is})|$ is uniformly bounded on compact subsets of D, dominated convergence applies and we may take the limit as $h\to 0$ under the integral sign. The result follows. □

We note that $\frac{\partial F}{\partial t}(r,t)$ exists at all points and is continuous there. However, the second derivative $\frac{\partial^2 F}{\partial t^2}(r,t)$, need not exist at all points. First, observe as before that $\frac{\partial^2|f'|}{\partial t^2}(re^{it})$ exists at all points except those where f' has a zero of odd order. But now this function need not be summable over a radial segment $[0,r_0e^{it}]$ which contains a zero of f'. This is unlike the situation for $\frac{\partial|f'|}{\partial t}(ue^{it})$.

To see this, consider the case $f'(z)=(z-a)g(z)$ where g is analytic and $g(a)\neq 0$ and $0<a<1$. We may assume that $z=a$ is the first zero of f' on the ray $t=0$. Then

$$\frac{\partial^2|f'|}{\partial t^2}(re^{it}) = |g(z)|\frac{\partial^2}{\partial t^2}|re^{it}-a| + 2\frac{\partial}{\partial t}|re^{it}-a|\frac{\partial|g|}{\partial t}(z) + |z-a|\frac{\partial^2|g|}{\partial t^2}(z).$$

The last two terms are summable over the interval $[0,a]$, so the first term is the crucial one. Let $|re^{it}-a|^2 = r^2-2ar\cos t+a^2 = L$. Then $\frac{\partial}{\partial t}|re^{it}-a| = \frac{ar\sin t}{L^{1/2}}$ and

$$\frac{\partial^2}{\partial t^2}|re^{it}-a| = \frac{Lar\cos t - a^2r^2\sin^2 t}{L^{3/2}}$$
$$= \frac{\{ar^3\cos t - a^2r^2(1+\cos^2 t)+a^3r\cos t\}}{L^{3/2}}.$$

Putting $t=0$, the right hand side (R.H.S.) equals $ar/(a-r)$ for $r<a$, and $ar/(r-a)$ for $r>a$. But this means that it is not summable over $[0,b]$ for any $b\ge a$. It follows that $\frac{\partial^2 F}{\partial t^2}(r,t)$ does not exist whenever $t=0$ and $r\ge a$ for this particular f.

Nevertheless, it remains true that if we exclude those rays $[0, e^{it})$ along which f' has zeros of odd order, a similar result to Lemma 3 holds. The formal statement is

Lemma 4. *If* e^{it} *is chosen as above, and* $0 < r < 1$*, then*

$$\frac{\partial^2 F}{\partial t^2}(r,t) = \int_0^r \frac{\partial^2 |f'|}{\partial t^2}(ue^{it})\, du.$$

Proof: Since $|f'(ue^{it})| > 0$ on the ray, it is clear that $\frac{\partial^2 |f'|}{\partial t^2}$ is a continuous function of u there. Moreover, $\frac{\partial^2 |f'|}{\partial s^2}(ue^{is})$ is a continuous function of both u and s provided $|s - t|$ is sufficiently small. The argument of Lemma 3 invoking the Mean Value Theorem, can now be applied to $\frac{\partial F}{\partial t}(r,t)$ just as before. $\square$

If f' has a zero of arbitrary odd order at a point, it will be found in a similar fashion that a sufficiently high order derivative of F will not exist along a part of the corresponding ray. It is seen that these are the only possibilities whereby F fails to be infinitely differentiable. The zeros of even order of f' on the other hand, do not cause problems for F. At all other points, F has derivatives of all orders. Indeed, it turns out that F has the further property of being (real) analytic on the complement of the segments identified above.

Recall a few facts about such analytic functions. A real function $g(x,y)$ is analytic at a point (x_0, y_0) if there is a neighbourhood U of the point (x_0, y_0) such that

$$g(x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{mn}(x - x_0)^m (y - y_0)^n, \quad (x,y) \in U,$$

where the series is absolutely convergent [**9**].

If f is an analytic function of z, then $|f(z)|$ is analytic in the real sense except at the odd zeros of f. It follows that $r = \sqrt{x^2 + y^2}$ is analytic except at $r = 0$. Also, $\cos t = x/r$, $\sin t = y/r$ are both analytic except at the origin. While $\frac{\partial g}{\partial x}$, $\frac{\partial g}{\partial y}$ are both analytic wherever g is, we note that $\frac{\partial g}{\partial r}(r,t) = \frac{\partial g}{\partial x}\cos t + \frac{\partial g}{\partial y}\sin t$ is not analytic at the origin even if g is analytic there. On the other hand, $\frac{\partial g}{\partial t}(r,t) = \frac{\partial g}{\partial x}(-r\sin t) + \frac{\partial g}{\partial y} r\cos t$ is analytic wherever g is.

These facts may now be applied to our situation. If f is an analytic function of z in D, then $|f'(re^{it}|$ is analytic except at zeros of f' of odd order. It is also true that $F(r,t) = \int_0^r |f'(ue^{it})|\, du$ is analytic at (r,t) if f' has no zeros of odd order on the line segment. However, F may not be analytic at $(0,0)$. Indeed, if f' is constant on the segment, then $F(r,t) = Cr$. We have seen that F cannot be analytic on $[be^{it}, e^{it})$, if f' has a zero of odd order at be^{it}, $0 < b < 1$. There are at most countably many such segments in the disc. Let H be the set on which F fails to be analytic; H is therefore a small set and we can say that F is analytic on $D \setminus H$. The same is true for $\frac{\partial F}{\partial t}$.

2.2. Integrability of $\frac{\partial^2 F}{\partial t^2}(r,t)$.

It will be necessary later to consider the integral of $\frac{\partial^2 F}{\partial t^2}(r,t)$ over $[-\pi,\pi]$, and to show that this exists. As the analysis above shows, the problem arises near a ray which contains an odd zero of f'. It is enough to consider f' replaced by $z-a$ where a is fixed, $0 < a < 1$ as before. It suffices to prove the following:

Lemma 5. *For each $r < 1$, $\frac{\partial F}{\partial t}(r,t)$ is absolutely continuous as a function of t.*

Proof: With f' as above we have

$$\frac{\partial F}{\partial t}(r,t) = \int_0^r \frac{au\sin t}{(a^2+u^2-2au\cos t)^{1/2}}\,du.$$

Let $L(t) = L(u,t) = u^2 - 2au\cos t + a^2$, as before. Suppose $\pi/2 > x > y > 0$ as we may, $0 < r \le a$, and consider

$$\begin{aligned}
\frac{\partial F}{\partial t}(r,t)\bigg|_{t=x} &- \frac{\partial F}{\partial t}(r,t)\bigg|_{t=y} \\
&= \int_0^r au\left(\frac{\sin x}{L(x)^{1/2}} - \frac{\sin y}{L(y)^{1/2}}\right)du \\
&= \int_0^r au\Big\{L(y)^{1/2}(\sin x - \sin y) \\
&\qquad\qquad - \sin y\big[L(x)^{1/2} - L(y)^{1/2}\big]\Big\}L(x)^{-1/2}L(y)^{-1/2}\,du.
\end{aligned}$$

Since $L(u,t) = (a-u)^2 + 4au\sin^2 t/2$, it is easy to see that

$$L(x)^{1/2} - L(y)^{1/2} \le 2a\Big(\sin(x/2) - \sin(y/2)\Big) \le a(x-y).$$

The numerator above is in absolute value less than or equal to

$$au\Big(L(y)^{1/2}(x-y) + a(x-y)\sin y\Big) = au(x-y)L(y)^{1/2} + a^2u(x-y)\sin y,$$

which gives rise to two terms in the integrand. If $a-u$ is small then $L(u,t)$ is small for small t and so the denominator is also small. Let us take the worst possible case $r = a$ and x, y both small. We split the integral into one over $(0, a-\epsilon)$ and one over $(a-\epsilon, a)$, where $\epsilon = \min\{ax, a/2\}$. Over $(0, a-\epsilon)$, $L(u,t) \geq (a-u)^2$, and the first term satisfies

$$(x-y)\int_0^{a-\epsilon} \frac{au}{L(u,x)^{1/2}}\,du \leq (x-y)a^2 \int_0^{a-\epsilon} \frac{du}{a-u} \leq a^2(x-y)\ln(1/\epsilon).$$

The second term is bounded by

$$a^2(x-y)\sin y \int_0^{a-\epsilon} \frac{du}{(a-u)^2} \leq a^2(x-y)\frac{\sin y}{\epsilon} \leq a(x-y)$$

since $\sin y/x \leq 1$. Passing to the integral over $(a-\epsilon, a)$, consider the second term first. Observe that for such u there is a constant C such that $L(u,t) \geq C^2(at)^2$ and therefore $a\sin y/L(y)^{1/2} \leq 1/C$. The integral is bounded by

$$\frac{a^2(x-y)}{C}\int_{a-\epsilon}^{a} \frac{du}{L(x)^{1/2}} \leq \frac{a^2(x-y)}{C}\int_{a-\epsilon}^{a} \frac{du}{Cax} = a^2(x-y)C^{-2}.$$

Clearly the first term contributes a like amount. The case $r > a$ is handled in the same way. We have shown that

$$\left| \frac{\partial F}{\partial t}(r,t)\Big|_{t=x} - \frac{\partial F}{\partial t}(r,t)\Big|_{t=y} \right| \leq (x-y)\Big(A + B\ln(1/x)\Big).$$

Since the function $t \to t\ln(1/t)$ is absolutely continuous for all t, the result follows. □

3. $f \in A^s_{pq}$

We now inquire what are the consequences for F of the assumption that $f \in A^s_{pq}$. Hereafter p' will denote the conjugate index of p.

3.1. The case $0 < s < 1$.

Theorem 1. *Suppose that $0 < s < 1$, $1 \leq p, q$. There is a constant $C = C(s,p,q)$ such that if $f \in A^s_{pq}$ then*

$$\int_0^1 (1-r^2)^{q(1-s)-1}\left(\int_{-\pi}^{\pi} |\nabla F(r,t)|^p\,dm\right)^{q/p} r\,dr \leq C||f||^q_A. \tag{2}$$

Proof: It is clear that if we replace ∇F in (2) by its first component, the inequality is trivially satisfied. Therefore to prove our theorem it suffices to show that for any r_0, $0 < r_0 < 1$,

$$(3)\quad \int_0^{r_0} (1-r^2)^{q(1-s)-1} \left(\int_{-\pi}^{\pi} 1/r^p \left| \frac{\partial F}{\partial t}(r,t) \right|^p dm \right)^{q/p} r\,dr \le C||f||_A^q.$$

The idea is to apply integration by parts with respect to r on the left hand side. This is valid if the inner integral is an absolutely continuous function of r which in turn is true if $\left|\frac{\partial F}{\partial t}(r,t)\right|$ is absolutely continuous in r, $r \le r_0$, and this follows from Lemma 3. We shall proceed formally and justify the operations later. The integrated term is $-\frac{(1-r^2)^{q(1-s)}}{2q(1-s)} \times \left(\int_{-\pi}^{\pi} 1/r^p \left|\frac{\partial F}{\partial t}\right|^p dm\right)^{q/p} \Big|_0^{r_0}$ and we first show this vanishes as $r_0 \to 1$. At the lower limit we may suppose that $f'(0) \ne 0$. Then for small r,

$$\begin{aligned} \left| 1/r \frac{\partial F}{\partial t}(r,t) \right| &= \left| 1/r \int_0^r \frac{\partial |f'|}{\partial t}(ue^{it})\,du \right| \\ &\le 1/r \int_0^r u|f''(ue^{it})|\,du \\ &\to 0, \end{aligned}$$

as $r \to 0$, uniformly in t, by Lemma 2. Consider the case as $r \to 1$. From Lemmas 2 and 3 we know that

$$1/r \left| \frac{\partial F}{\partial t}(r,t) \right| = 1/r \left| \frac{\partial}{\partial t} \int_0^r |f'(ue^{it})|\,du \right| \le 1/r \int_0^r u|f''(ue^{it})|\,du,$$

whence by Minkowski's inequality,

$$(4)\quad \begin{aligned} \left(\int_{-\pi}^{\pi} 1/r^p \left| \frac{\partial F}{\partial t}(r,t) \right|^p dm \right)^{1/p} &\le \left(\int_{-\pi}^{\pi} \left(\int_0^r |f''(ue^{it})|\,du \right)^p dm \right)^{1/p} \\ &\le \int_0^r \left(\int_{-\pi}^{\pi} |f''(ue^{it})|^p\,dm \right)^{1/p} du. \end{aligned}$$

From the definition it follows that $\int_r^1 (1-u^2)^{q(2-s)-1} M_p(f'',u)^q\,du = o(1)$, as $r \to 1$, from which $(1-r^2)^{q(2-s)} M_p(f'',r)^q = o(1)$, as $r \to 1$. This gives

$$\int_0^r M_p(f'',u)\,du = o(1) \int_0^r \frac{du}{(1-u^2)^{2-s}} = \frac{o(1)}{(1-r^2)^{1-s}}, \quad (r \to 1),$$

and the claim now follows from (4). Returning to the main thread, the left hand side (L.H.S.) of (3) becomes

$$\frac{1}{2q(1-s)}\int_0^{r_0}(1-r^2)^{q(1-s)}\frac{\partial}{\partial r}\left(1/r^p\int_{-\pi}^{\pi}\left|\frac{\partial F}{\partial t}\right|^p\,dm\right)^{q/p}r\,dr,$$

which in turn equals

$$\begin{aligned}&\frac{1}{2p(1-s)}\int_0^{r_0}(1-r^2)^{q(1-s)}\frac{1}{r^q}\left(\int_{-\pi}^{\pi}\left|\frac{\partial F}{\partial t}\right|^p\,dm\right)^{-1+q/p}\\&\qquad\qquad\times\left(p\int_{-\pi}^{\pi}\left|\frac{\partial F}{\partial t}\right|^{p-1}\frac{\partial}{\partial r}\left|\frac{\partial F}{\partial t}\right|\,dm\right)r\,dr\\&-\frac{q}{2q(1-s)}\int_0^{r_0}(1-r^2)^{q(1-s)}r^{-q-1}\left(\int_{-\pi}^{\pi}\left|\frac{\partial F}{\partial t}\right|^p\,dm\right)^{q/p}r\,dr\\&=T_1+T_2.\end{aligned}$$

We notice that $T_2<0$ whereas the sum is positive. We can therefore discard T_2 and the L.H.S. is bounded by $|T_1|$ which is less than

$$\begin{aligned}\frac{1}{2(1-s)}\int_0^{r_0}(1-r^2)^{q(1-s)}r^{-q}&\left(\int\left|\frac{\partial F}{dt}\right|^p\,dm\right)^{-1+q/p}\\&\times\left(\int\left|\frac{\partial F}{\partial t}\right|^{p-1}\left|\frac{\partial}{\partial r}\left|\frac{\partial F}{\partial t}\right|\right|\,dm\right)r\,dr.\end{aligned}$$

Write $K=\frac{1}{2(1-s)}$. Using Lemmas 3, 1 and 2 again, we can replace the R.H.S. by

$$\begin{aligned}K\int_0^{r_0}(1-r^2)^{q(1-s)}\frac{1}{r^q}&\left(\int\left|\frac{\partial F}{\partial t}\right|^p\,dm\right)^{-1+q/p}\\&\times\left(\int\left|\frac{\partial F}{\partial t}\right|^{p-1}r|f''(re^{it})|\,dm\right)r\,dr.\end{aligned}\tag{5}$$

We apply Hölder's inequality with indices p', p to the second of the inner integrals to obtain

$$\int_{-\pi}^{\pi}\left|\frac{\partial F}{\partial t}\right|^{p-1}|f''(re^{it})|\,dm\le\left(\int_{-\pi}^{\pi}\left|\frac{\partial F}{\partial t}\right|^p\,dm\right)^{1/p'}M_p(f'',r),$$

which allows us to replace (5) on the R.H.S. by

$$K\int_0^{r_0}(1-r^2)^{q(1-s)}r^{-q+1}\left(\int_{-\pi}^{\pi}\left|\frac{\partial F}{\partial t}\right|^p dm\right)^{(q-1)/p} M_p(f'',r)r\,dr,$$

which in turn is equal to

$$K\int_0^{r_0}(1-r^2)^{q(1-s)}\left(\int_{-\pi}^{\pi}r^{-p}\left|\frac{\partial F}{\partial t}\right|^p dm\right)^{(q-1)/p} M_p(f'',r)r\,dr.$$

Next we write $(1-r^2)^{q(1-s)}=(1-r^2)^{(q-1)(1-s)-1/q'}(1-r^2)^{2-s-1/q}$, and apply Hölder's Inequality again, this time with indices q', q, to replace the R.H.S. by K times

$$\left(\int_0^{r_0}(1-r^2)^{q(1-s)-1}\left(\int\frac{1}{r^p}\left|\frac{\partial F}{\partial t}\right|^p dm\right)^{q/p} r\,dr\right)^{1/q'} \times\left(\int_0^{r_0}(1-r^2)^{q(2-s)-1}M_p(f'',r)^q r\,dr\right)^{1/q}.$$

Summing up, we can say that the L.H.S. of (3) is bounded by this last expression. However, a cancellation is now possible after which we conclude that

$$\left(\int_0^{r_0}(1-r^2)^{q(1-s)-1}\left(\int\frac{1}{r^p}\left|\frac{\partial F}{\partial t}\right|^p dm\right)^{q/p} r\,dr\right)^{1/q} \le K\left(\int_0^{1}(1-r^2)^{q(2-s)-1}M_p(f'',r)^q r\,dr\right)^{1/q}.$$

The R.H.S. is bounded by a constant times $\|f\|_A$. Since this holds for all $r_0<1$, the desired inequality follows and proves that $F(r,t)$ satisfies the stipulated condition. $\square$

Remarks. (1) The first point to be validated here is the differentiation under the integral sign in the T_1 term. For $x\ge 0$, $y\ge 0$, $p\ge 1$, the inequality $|x^p-y^p|\le p|x-y|(x^{p-1}+y^{p-1})$, holds; see Section 41 of [**13**]. This inequality together with Lemma 2 gives for all $r,s\le r_0$, $r\ne s$ and all t,

$$\begin{aligned}\left|\left|\frac{\partial F}{\partial t}(r,t)\right|^p - \left|\frac{\partial F}{\partial t}(s,t)\right|^p\right| &\leq p\left|\frac{\partial F}{\partial t}(r,t) - \frac{\partial F}{\partial t}(s,t)\right|\left\{\left|\frac{\partial F}{\partial t}(r,t)\right|^{p-1} + \left|\frac{\partial F}{\partial t}(s,t)\right|^{p-1}\right\}\\ &= p\left|\int_s^r \frac{\partial |f'|}{\partial t}(ue^{it})\,du\right|\left\{\left|\frac{\partial F}{\partial t}(r,t)\right|^{p-1} + \left|\frac{\partial F}{\partial t}(s,t)\right|^{p-1}\right\}\\ &\leq p\left|\int_s^r u|f''(ue^{it})|\,du\right|\left\{\left|\frac{\partial F}{\partial t}(r,t)\right|^{p-1} + \left|\frac{\partial F}{\partial t}(s,t)\right|^{p-1}\right\}.\end{aligned}$$

Divide across by $r-s$ and let $H(r,s,t) = \frac{1}{r-s}\int_s^r u|f''(ue^{it})|\,du$. It is clear that $B(t) = \sup\{H(r,s,t);\, r,s \leq r_0\}$ is a bounded function on $(-\pi,\pi)$. Equally, the term in curly brackets is also a bounded function of (r,t) for $r \leq r_0$. An appeal to dominated convergence is therefore valid, the limit may be taken as $s \to r$, and differentiation under the integral sign is justified.

(2) A further consideration needs to be dealt with. Since $\left|\frac{\partial F}{\partial t}(r,t)\right|$ is absolutely continuous as a function of r (see Lemma 3), it follows that $\frac{\partial}{\partial r}\left|\frac{\partial F}{\partial t}\right|(r,t)$ exists a.e. r for all t.

3.2. The case $s = 1$.

For the case $s = 1$, we introduce the Laplacian of F, namely

$$\Delta F(r,t) = \frac{\partial^2 F}{\partial r^2} + 1/r\frac{\partial F}{\partial r} + 1/r^2\frac{\partial^2 F}{\partial t^2}.$$

In this case our results are less general; we comment on this further below. We shall require that $p = 1$ in order to get a result similar to that above. To set the scene, we first present a very simple special case.

Theorem 2. *There is a constant C such that for all $f \in A^1_{11}$*

$$\int_0^1\int_{-\pi}^{\pi} |\Delta F(r,t)|\,dm\,r\,dr \leq C||f||_A.$$

Proof: We know that F is analytic except on the set H. An important property of F is that it is subharmonic [**4**], and therefore $\Delta F(r,t) \geq 0$ where this exists. Fix $r < 1$, let $[be^{it}, e^{it})$ be a line segment which intersects $H \cap D(0,r)$. Enclose this segment in a narrow strip of width ϵ.

These strips are at most finite in number. If necessary, enclose the origin also in a disc of radius ϵ. Let $G(\epsilon)$ be that part of $D(0,r)$ with these subsets excluded. Green's Theorem [**9**] may be applied to F over the domain $G(\epsilon)$:

$$\pi \iint_{G(\epsilon)} \Delta F(u,t)\, dm_2 = \int_{\partial G(\epsilon)} \frac{\partial F}{\partial n}\, ds.$$

On the side of each strip the outward normal derivative is $\pm\frac{\partial F}{\partial t}$. By continuity, the integrals over these sides cancel pairwise when $\epsilon \to 0$. In the limit we get

$$\int_0^r \int_{-\pi}^{\pi} \Delta F(u,t)\, dm\, u\, du = r \int_{-\pi}^{\pi} \frac{\partial F}{\partial r}\, dm = r \int_{-\pi}^{\pi} |f'(re^{it})|\, dm.$$

But $f'(re^{it}) = f'(0) + e^{it} \int_0^r f''(ue^{it})\, du$ which yields, on taking absolute values,

$$\int_{-\pi}^{\pi} |f'(re^{it})|\, dm \le |f'(0)| + \int_0^r \int_{-\pi}^{\pi} |f''(ue^{it})|\, dm\, du.$$

We now let $r \to 1$ and deduce that there is a constant C such that

$$\int_0^1 \int_{-\pi}^{\pi} |\Delta F(r,t)|\, dm\, r\, dr \le C||f||_A.$$

This completes the proof. $\square$

We can progress beyond this case to a limited extent.

Theorem 3. *Suppose that $1 \le q < 2$, and that $f \in A^1_{1q}$. Then there exists $C = C_q$ such that*

$$\int_0^1 (1-r^2)^{q-1} \left(\int_{-\pi}^{\pi} |\Delta F(r,t)|\, dm \right)^q r\, dr < C||f||_A^q.$$

Proof: Since $\Delta F(r,t) \ge 0$, it is enough to consider each of the three terms in turn. From Lemma 5

$$\int_{-\pi}^{\pi} \frac{\partial^2 F}{\partial t^2}\, dt = \left.\frac{\partial F}{\partial t}\right|_{-\pi}^{\pi} = 0, \quad 0 < r < 1.$$

Next we have, using Lemmas 1 and 2,

$$\begin{aligned}
&\left|\int_0^1 (1-r^2)^{q-1}\left(\int_{-\pi}^{\pi}\frac{\partial^2 F}{\partial r^2}\,dm\right)^q r\,dr\right| \\
&\qquad\le \int_0^1 (1-r^2)^{q-1}\left(\int_{-\pi}^{\pi}\left|\frac{\partial|f'|}{\partial r}(re^{it})\right|\,dm\right)^q r\,dr \\
&\qquad\le \int_0^1 (1-r^2)^{q-1}\left(\int_{-\pi}^{\pi}|f''(re^{it})|\,dm\right)^q r\,dr \\
&\qquad\le C||f||_A^q.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\int_0^1 (1-r^2)^{q-1}\left(\int \frac{\partial F}{\partial r}\frac{dm}{r}\right)^q r\,dr \\
&\qquad= \int_0^1 (1-r^2)^{q-1}\left(\int |f'(re^{it})|\frac{dm}{r}\right)^q r\,dr \\
&\qquad\le \int_0^1 (1-r^2)^{q-1}\left(\int \left[|f'(0)|+\int_0^r |f''(ue^{it})|\,du\right]\frac{dm}{r}\right)^q r\,dr \\
&\qquad\le 2^{q-1}\left\{||f||_A^q+\int_0^1 (1-r^2)^{q-1}\left(\int_0^r M_1(f'',u)\frac{du}{r}\right)^q r\,dr\right\} \\
&\qquad\le 2^{q-1}||f||_A^q+2^{q-1}\int_0^1 (1-r^2)^{q-1}M_1(f'',r)^q r\,dr \\
&\qquad\le C_q||f||_A^q.
\end{aligned}$$

Above, we used the fact that $\int_0^1 (1-u)^{q-1}u^{-q/2}\,du = B(q,1-q/2)$, is finite. Putting all these together the result follows. □

Remarks. (1) To see that the restriction on q is necessary, take $f(z)=z$ so that $|f'(z)|\equiv 1$. Then $F(r,t)=r$ everywhere in the disc. It follows that $\Delta F(r,t)=1/r$ and the integral on the left above diverges if $q\ge 2$.

(2) We needed $p=1$ in order to be able to use the subharmonicity of F which was crucial to our argument.

(3) The limitation on q noted above, holds for all p. We cannot expect a general theorem along the lines of Theorem 1 to hold if we replace ∇F by ΔF, as the example $f(z)=z$ above shows.

4. The boundary function

4.1. The case $0 < s < 1$.

We recall that $F(t)$, the boundary function of $F(r,t)$, exists, finite or infinite, for all t. We have shown that if $0 < s < 1$, $1 \leq p, q < \infty$, and if $f \in A^s_{pq}$, then

$$\int_0^1 (1-r^2)^{q(1-s)-1} \left(\int_{-\pi}^{\pi} |\nabla F(r,t)|^p \, dm \right)^{q/p} r \, dr \leq C||f||^q_A. \tag{6}$$

Apart from its intrinsic interest, this result can be used to show that $F(t)$ is in the space B^s_{pq} on the circle; that in fact

$$||F||_B \leq C||f||_B.$$

This is a direct analogue of Beurling's result which he used to prove his main result on capacity. The method is standard and has been used by Hardy and Littlewood; see e.g. [**14**].

Theorem 4. *Suppose that* $0 < s < 1$, $1 \leq p, q < \infty$. *There is a constant* $C = C(p,q,s)$ *such that if* $f \in A^s_{pq}$, *then* $F \in B^s_{pq}$ *and*

$$\int_{-\pi}^{\pi} |t|^{-1-sq} \left(\int_{-\pi}^{\pi} |F(v+t) - F(v)|^p \, dm(v) \right)^{q/p} dm(t) \leq C||f||^q_B. \tag{7}$$

Proof: Our first task is to show that $F \in L^p(T)$. Note that $f \in A^s_{pq}$ implies that $(1-r^2)^{q(1-s)} M_p(f', r)^q = o(1)$, as $r \to 1$, or $M_p(f', r) = o(1)(1-r)^{s-1}$. Fix $r < 1$. It follows from the definition of F and from Minkowski's Inequality that

$$\left(\int_{-\pi}^{\pi} F(r,t)^p \, dm \right)^{1/p} \leq \int_0^r M_p(f', u) \, du \leq C,$$

where C is independent of r. Letting $r \to 1$, the conclusion follows from monotone convergence.

We write

$$\begin{aligned} F(v+t) - F(v) = {} & [F(v+t) - F(r, v+t)] \\ & + [F(r, v+t) - F(r,v)] + [F(r,v) - F(v)] \end{aligned} \tag{8}$$

where r, $0 < r < 1$, is at our disposal. Choose r such that $1 - r = t/\pi$. Observe that $F(v+t) - F(r, v+t) = \int_r^1 |f'(we^{i(v+t)}| \, dw$. Taking the L^p-norm with respect to the variable v we have

$$||F(\dot{v}+t) - F(r, \dot{v}+t)||_p \leq \int_r^1 M_p(f', w) \, dw.$$

Here and hereafter a dot over a variable signifies integration with respect to that variable.

We proceed to estimate the $||\cdot||_B$ of each term on R.H.S. of (8). It suffices to consider the integral from 0 to π only, and we shall use dt here rather than $dm(t)$.

$$\left(\int_0^\pi t^{-1-sq}||F(\cdot)-F(r,\cdot)||_p^q\,dt\right)^{1/q} \leq \left(\int_0^\pi t^{-1-sq}\left(\int_r^1 M_p(f',w)\,dw\right)^q dt\right)^{1/q}.$$

Change the variable, letting $w = 1-u/\pi$. Put $H(u) = M_p(f',(1-u/\pi))$. Since $0 \leq u \leq \pi$, let $H(u) = 0$ outside this range. We now apply the following inequality of Hardy [**14**], [**8**]: suppose $q \geq 1$, $l > 0$, $0 \leq g \in L(0,\infty)$, then

$$\left(\int_0^\infty \left(\int_0^x g(y)\,dy\right)^q x^{-l-1}\,dx\right)^{1/q} \leq q/l\left(\int_0^\infty \Big(xg(x)\Big)^q x^{-l-1}\,dx\right)^{1/q}.$$

We apply this to the case where $g = H/\pi$, $l = sq$, and obtain

$$\begin{aligned}
&\left\{\int_0^\pi \frac{1}{t^{1+sq}}\left(\int_0^t H(u)\frac{du}{\pi}\right)^q dt\right\}^{1/q}\\
&\quad\leq \frac{1}{s}\left\{\int_0^\pi\left(\frac{uH(u)}{\pi}\right)^q u^{-1-sq}\,du\right\}^{1/q}\\
&\quad\leq \frac{1}{s}\left\{\frac{1}{\pi^q}\int_0^\pi u^{(1-s)q-1}M_p(f',(1-u/\pi))^q\,du\right\}^{1/q}\\
&\quad= C\left\{\int_0^1 (1-w)^{(1-s)q-1}M_p(f',w)^q\,dw\right\}^{1/q}\\
&\quad\leq C||f||_A.
\end{aligned}$$

Next, consider the second term $F(r,v+t)-F(r,v) = \int_0^t \frac{\partial F}{\partial\theta}(r,v+\theta)\,d\theta$. Again, by Minkowski's inequality we have

$$\begin{aligned}
||F(r,\dot{v}+t)-F(r,\dot{v})||_p &\leq \int_0^t \left\|\frac{\partial F}{\partial\theta}(r,\dot{v}+\theta)\right\|_p d\theta\\
&= t\left\|\frac{\partial F}{\partial\theta}(r,\dot{\theta})\right\|_p.
\end{aligned}$$

Letting $t = \pi(1-r)$,

$$J = \left\{ \int_0^{\pi} t^{-1-sq} \left(\int_{-\pi}^{\pi} |F(r, v+t) - F(r, v)|^p \, dm(v) \right)^{q/p} dt \right\}^{1/q},$$

the last observation gives

$$\begin{aligned} J &\leq \left\{ \int_0^{\pi} t^{-1-sq} t^q \left\| \frac{\partial F}{\partial \theta}(r, \dot{\theta}) \right\|_p^q dt \right\}^{1/q} \\ &\leq \left\{ \pi^{q(1-s)} \int_0^1 (1-r)^{q(1-s)-1} \left\| \frac{\partial F}{\partial \theta}(r, \dot{\theta}) \right\|_p^q dr \right\}^{1/q} \\ &\leq C||f||_A \leq C||f||_B, \end{aligned}$$

by Theorem 1, and the equivalence of the norms on f. The third term on the R.H.S. of (8), $F(r, v) - F(v)$, is dealt with in exactly the same way as the first term, and (7) now follows. □

4.2. The case $s = 1$.

We now consider the case $s = 1$ and ask which if any of the results above hold for F.

Theorem 5. *Suppose* $1 \leq q < 2$. *There exists a constant* $C = C_q$, *such that if* $f \in A^1_{1q}$ *then* $F \in B^1_{1q}$, *and*

$$||F||_B \leq C||f||_A.$$

Proof: We recall that $||f||_A^q = ||f||_1^q + \int_0^1 (1-r^2)^{q-1} \left(\int |f''(re^{it})| \, dm \right)^q r \, dr < \infty$. Since $A^1_{1q} \subset A^s_{1q}$ for any $s < 1$, the fact that $F \in L^1(T)$ follows from the proof in Theorem 4. It is enough to show that

$$\int_0^{\pi} t^{-1-q} \left(\int_0^{\pi} |F(v+t) + F(v-t) - 2F(v)| \, dm(v) \right)^q dt \leq C||f||_A^q.$$

Recall that $\Delta_t^2 F(v-t) = F(v+t) + F(v-t) - 2F(v)$. For convenience, write $L_r(v) = F(v) + F(2r-1, v) - 2F(r, v)$. This allows us to write

$$\begin{aligned} \Delta_t^2 F(v-t) = L_r(v+t) + L_r(v-t) - 2L_r(v) \\ - \Delta_t^2 F(2r-1, v-t) + 2\Delta_t^2 F(r, v-t). \end{aligned}$$

The first three terms are all of the same kind, and the last two terms are also of the same kind. We choose $1 - r = t/\pi$. Take a typical term in the first group, $L_r(v)$, and write it as

$$
\begin{aligned}
&F(v) + F(2r-1, v) - 2F(r, v) \\
&\qquad = \int_r^1 |f'(ue^{iv})|\,du - \int_{2r-1}^r |f'(ue^{iv})|\,du \\
&\qquad = \int_0^{1-r} \left\{|f'\big((r+u)e^{iv}\big)| - |f'\big((2r-1+u)e^{iv}\big)|\right\} du,
\end{aligned}
$$

whence

$$
\begin{aligned}
|L_r(v)| &\le \int_0^{1-r} \Big\{\big|f'\big((r+u)e^{iv}\big) - f'\big((2r-1+u)e^{iv}\big)\big|\Big\}\, du \\
&= \int_0^{1-r} \left|\int_{2r-1+u}^{r+u} f''(we^{iv})\,dw\right| du \\
&\le \int_0^{1-r} \int_{2r-1+u}^{r+u} \big|f''(we^{iv})\big|\, dw\, du.
\end{aligned}
$$

The region of integration is a parallelogram in the uw plane. Let us now switch the order of integration. If $2r - 1 < w < r$, then the limits for u are $0 < u < w - (2r-1)$, while if $r < w < 1$, they are $w - r < u < 1 - r$. The last displayed double integral now becomes

$$
\begin{aligned}
&\int_{2r-1}^r \left(\int_0^{w-(2r-1)} |f''(we^{iv})|\,du\right) dw + \int_r^1 \left(\int_{w-r}^{1-r} |f''(we^{iv})|\,du\right) dw \\
&\qquad = \int_{2r-1}^r \big(w - 2r + 1\big)|f''(we^{iv})|\,dw + \int_r^1 (1-w)|f''(we^{iv})|\,dw.
\end{aligned}
$$

First taking the L^1 norm with respect to v, we get

$$
||L_r(\dot{v})||_1 \le \int_{2r-1}^r |w - 2r + 1| M_1(f'', w)\,dw + \int_r^1 (1-w) M_1(f'', w)\,dw.
$$

Next we calculate the outer norm of $\|L_r(\dot v)\|$ and apply Minkowski's Inequality to get

$$\begin{aligned}
&\left(\int_0^\pi t^{-1-q}\|L_r(\dot v)\|_1^q\,dt\right)^{1/q}\\
&\qquad\le \left(\int_0^\pi \frac{1}{t^{1+q}}\left(\int_{2r-1}^r |w-2r+1|M_1(f'',w)\,dw\right)^q dt\right)^{1/q}\\
&\qquad\quad + \left(\int_0^\pi \frac{1}{t^{1+q}}\left(\int_r^1 (1-w)M_1(f'',w)\,dw\right)^{1/q} dt\right)^{1/q}\\
&\qquad = T_1+T_2.
\end{aligned}$$

Starting with T_1, we change the variable. Let $w = 2r-1+y/\pi$ and note that $y=0$ when $w=2r-1$ and $y=t$ when $w=r$. Then

$$T_1 \le \left(\int_0^\pi t^{-1-q}\left(1/\pi^2\int_0^t yM_1(f'',2r-1+y/\pi)\,dy\right)^q dt\right)^{1/q}.$$

Applying Hardy's Inequality to the R.H.S. with $g(y) = yM_1(f'',2r-1+\ y/\pi)$, $l=q$, gives

$$\begin{aligned}
T_1 &\le \left(\pi^{-2q}\int_0^\pi t^{2q}M_1^q(f'',2r-1+t/\pi)t^{-1-q}\,dt\right)^{1/q}\\
&\le \pi^{-2}\left(\int_0^\pi t^{q-1}M_1^q(f'',r)\,dt\right)^{1/q}\\
&= \pi^{-2}\left(\pi^q\int_0^1 (1-r)^{q-1}M_1^q(f'',r)\,dr\right)^{1/q}\\
&\le C||f||_A,
\end{aligned}$$

for some absolute constant C. A similar argument applies to T_2. This time we let $w = 1 - y/\pi$ and note that $y = t$ when $w = r$. We have

$$\begin{aligned} T_2 &\le \left(\int_0^\pi t^{-1-q} \left(1/\pi^2 \int_0^t y M_1(f'', 1 - y/\pi)\, dy \right)^q dt \right)^{1/q} \\ &\le \left(\pi^{-2q} \int_0^\pi t^{2q} M_1^q(f'', 1 - t/\pi) t^{-1-q}\, dt \right)^{1/q} \\ &= 1/\pi \left(\int_0^1 (1-r)^{q-1} M_1^q(f'', r)\, dr \right)^{1/q} \le C||f||_A. \end{aligned}$$

This takes care of the three terms of the first type. Next consider a term of the second type. Lemma 5 allows us to apply the Fundamental Theorem of Calculus:

$$\begin{aligned} \Delta_t^2 F(r, v - t) &= \int_0^t \frac{\partial F}{\partial x}(r, v + x)\, dx - \int_0^t \frac{\partial F}{\partial x}(r, v - t + x)\, dx \\ &= \int_0^t \left(\int_0^t \frac{\partial}{\partial y} \left(\frac{\partial F}{\partial x}(r, v - t + x + y) \right) dy \right) dx \\ &= \int_0^t \left(\int_0^t \frac{\partial^2 F}{\partial y^2}(r, v - t + x + y)\, dy \right) dx, \end{aligned}$$

due to the fact that differentiation with respect to x is the same as differentiation with respect to y. From this it follows that

$$\begin{aligned} ||\Delta_t^2 F(r, \dot{v} - t)||_1 &\le \int_0^t \int_0^t \left\| \frac{\partial^2 F}{\partial y^2}(r, \dot{v} - t + x + y) \right\|_1 dy\, dx \\ &= t^2 \left\| \frac{\partial^2 F}{\partial v^2}(r, \dot{v}) \right\|_1 . \end{aligned}$$

The last step is to take the outer norm involving integration with respect to t. In doing this we invoke Theorem 3 which was a result about the A-norm of F. We claim that

$$\begin{aligned} \left(\int_0^\pi t^{-1-q} \|\Delta_t^2 F(r, \dot{v} - t)\|_1^q\, dt \right)^{1/q} &\le \left(\int_0^\pi t^{-1-q} t^{2q} \left\| \frac{\partial^2 F}{\partial v^2}(r, \dot{v}) \right\|_1^q dt \right)^{1/q} \\ &= \left(\pi^q \int_0^1 (1-r)^{q-1} \left\| \frac{\partial^2 F}{\partial v^2}(r, \dot{v}) \right\|_1^q dr \right)^{1/q} \\ &\le C||f||_A. \end{aligned}$$

To see this, we use the fact that

$$\left\|\frac{\partial^2 F}{\partial v^2}(r,\dot{v})\right\|_1 \leq r^2||\Delta F(r,\cdot)||_1 + r\left\|\frac{\partial F}{\partial r}(r,\cdot)\right\|_1 + r^2\left\|\frac{\partial^2 F}{\partial r^2}(r,\cdot)\right\|_1.$$

Our claim will follow if we can show that each of the three terms on the right hand side satisfies the last inequality in the previous display. But we have already seen in Theorem 3 that each is bounded by $C||f||_A$ for some absolute constant C. We know that $||f||_A \equiv ||f||_B$. Putting all these considerations together the result follows for this case. □

Remark. If $q = 1$, then $f \in A^1_{11}$ and it is easy to see that this implies that $f' \in H^1$. This implies further [**7**] that f is absolutely continuous on T. We have further

Proposition 1. *If $f \in A^1_{11}$, then F is absolutely continuous on T.*

Proof: First, observe that it follows from the Fejér-Riesz Inequality [**7**] that $F(t)$ is finite everywhere:

$$\int_{-1}^{1} |f'(ue^{it})|\,du = F(0) + F(\pi) \leq 1/2\int_0^{2\pi} |f'(e^{it})|\,dt.$$

We know from the previous theorem that $F \in B^1_{11}$. Hence the Riesz projection $PF \in A^1_{11}$ and is therefore absolutely continuous. Since a similar remark applies to $(I-P)F$, the result follows. □

5. The exceptional set

5.1. Alternative characterizations.

Suppose that f is a function on T with Fourier series $f(e^{it}) \sim \sum_{-\infty}^{\infty} a_n e^{int}$. We shall denote the fractional derivative of order s of f by $D^s f$ where

$$D^s f(e^{it}) \sim \sum_{-\infty}^{\infty} (in)^s a_n e^{int}.$$

The Bessel potential space L^p_s is defined to be the space of functions whose fractional derivatives of order s are in L^p; see Appendix 2.6 of [**12**]. We shall make use of the following inclusion [**14**]:

$$B^s_{pp} \subset L^p_s, \quad 1 < p \leq 2.$$

By the Hausdorff-Young theorem [**7**], we have for such p,

$$\left(\sum_{-\infty}^{\infty} n^{sp'}|a_n|^{p'}\right)^{1/p'} \leq ||D^s f||_p. \tag{9}$$

In [**4**], Beurling considered functions f on the unit circle T which belong to $B_2^{1/2}$, and described the set E of points e^{ix} for which the symmetric derivative,

$$\lim_{h\to 0} \frac{1}{2h} \int_{x-h}^{x+h} f(t)\, dt,$$

fails to exist. He proved the remarkable result that the set E coincides with two other sets of points, (i) the set of points on T for which the Fourier series of f diverges, and (ii) the set of points e^{ix} for which $\lim_{r\to 1} f(re^{ix})$, the radial limit of the Poisson integral of f on the unit disc D, fails to exist. Beurling showed that this set E has logarithmic capacity zero. The first of the following lemmas, due to Fejér, was used by Beurling in his work and is stated here just for comparison; the second is easily proved.

Lemma 6. *Suppose that* $g(z) = \sum_1^\infty b_n z^n$, $|z| < 1$, *and that* $\sum_1^\infty n|b_n|^2 < \infty$. *Suppose also that the radial limit,* $\lim_{r\to 1} g(r)$, *exists and equals* B. *Then* $\sum_1^\infty b_n = B$. *Conversely, if the series converges then the radial limit exists and they have the same value.*

Lemma 7. *Suppose that* $s_n \geq 0$ *for all* $n \geq 1$ *and that* $\sum_1^\infty s_n$ *converges. Then* $1/n \sum_1^n k s_k \to 0$ *as* $n \to \infty$.

Suppose that $a_0 = 0$ for convenience. It is clear that

$$\frac{1}{2h} \int_{x-h}^{x+h} f(t)\, dt = \sum_{-\infty}^{\infty} a_n \frac{\sin nh}{nh} e^{inx}. \tag{10}$$

We remark that the series on the right converges absolutely if $f \in L^p$, $1 < p$, which will be true for the cases we have in mind. Suppose now that $f \in B_{pp}^s$ where $1 < p \leq 2$ and $s \geq 1/p$. Suppose that f has the property that $\lim_{r\to 1} f(re^{it})$ exists and equals B. We can show that the series $\sum_{-\infty}^\infty a_n e^{int}$ also converges to the same value B and conversely. Further, the symmetric derivative of f exists at e^{it} and has the value B also. Since the space B_{pp}^s contains the conjugate of each of its members, it suffices to consider analytic $f \in A_{pp}^s$.

Theorem 6. *Suppose that* $1 < p \leq 2$, $f \in A_{pp}^s$ *for some* $s \geq 1/p$ *and* $f(z) = \sum_1^\infty a_n z^n$. *Suppose also that the radial limit,* $\lim_{r\to 1} f(re^{ix})$, *exists and equals* B. *Then* $\sum_1^\infty a_n e^{inx}$ *converges to the same sum* B. *Conversely, if the series converges then the radial limit exists and they have the same value.*

Proof: From (9) we know that $\sum_1^\infty n^{sp'}|a_n|^{p'} < \infty$. Consider the difference

$$s_n(e^{ix}) - f(re^{ix}) = \sum_{k=1}^{n} a_k e^{ikx}(1-r^k) - \sum_{n+1}^{\infty} a_k r^k e^{ikx} = T_1 + T_2.$$

Using the fact that $1 - r^k \le k(1-r)$ we have, by Hölder's Inequality and Lemma 8,

$$\begin{aligned}
|T_1| &\le \sum_1^n k|a_k|(1-r) = (1-r)\sum_1^n k^{s+1/p'}|a_k|k^{1-s-1/p'} \\
&\le (1-r)\left(\sum_1^n k^{1+sp'}|a_k|^{p'}\right)^{1/p'}\left(\sum_1^n k^{1-sp}\right)^{1/p} \\
&\le (1-r)\{o(n)\}^{1/p'} n^{(2-sp)/p} \\
&\le (1-r)n^{1+1/p-s}o(1).
\end{aligned}$$

Next, consider the second term and use Hölder's Inequality again,

$$\begin{aligned}
|T_2| &\le \sum_{n+1}^{\infty} |a_k|r^k = \sum_{n+1}^{\infty} k^s|a_k|k^{-s}r^k \\
&\le \left(\sum_{n+1}^{\infty} k^{sp'}|a_k|^{p'}\right)^{1/p'}\left(\sum_{n+1}^{\infty} r^{kp}k^{-sp}\right)^{1/p} \\
&\le o(1)n^{-s}r^n(1-r)^{-1/p}.
\end{aligned}$$

Both terms can be made small provided that

(a) $\qquad (1-r)n^{1+1/p-s} \le M, \quad \text{say;}$

(b) $\qquad (1-r)n^{sp} \ge 1.$

Comparing the index sp with $1 + 1/p - s$, we see that if $(1-r)n^{sp} = 1$ then (a) is satisfied if and only if $sp \ge 1$. It follows therefore, that if $sp \ge 1$, then the difference can be made arbitrarily small as $n \to \infty$, or, equivalently, as $r \to 1$. Consequently, the first statement of the theorem follows. But the same argument applies equally to the second statement, and the theorem is proved. □

It is a pleasant fact that the same argument can be readily adapted to prove the second equivalence, that the symmetric derivative exists at a point if and only if the Fourier series of f converges at that point.

Theorem 7. *Suppose that p, s, f are as above. If $\sum_1^\infty a_k e^{ikx}$ converges with sum B, then $\lim_{h\to 0}\sum_1^\infty a_k e^{ikx}\frac{\sin kh}{kh}$ exists and equals B, and conversely.*

Proof: We consider a difference as before with $f(re^{ix})$ replaced by $\sum_1^\infty a_k e^{ikx}\frac{\sin kh}{kh}$, and write it as a sum of two terms T_1+T_2. Choose n initially so that nh is bounded. For T_2 we have

$$|T_2| \le 1/h\sum_{n+1}^{\infty} k^s|a_k|k^{-1-s} \le 1/h\left(\sum_{n+1}^{\infty} k^{sp'}|a_k|^{p'}\right)^{1/p'}\left(\sum_{n+1}^{\infty} k^{-p-sp}\right)^{1/p}$$
$$\le o(1)n^{-1-s+1/p}/h.$$

To estimate T_1, note that there exists a constant M such that $|1-\sin x/x| \le Mx^2$, for $|x|\le 1$. By a familiar argument,

$$\begin{aligned}
|T_1| &\le \sum_1^n |a_k|\left|1-\frac{\sin kh}{kh}\right| \le Mh^2\sum_1^n k^{s+1/p'}|a_k|k^{2-s-1/p'}\\
&\le Mh^2\left(\sum_1^n k^{sp'+1}|a_k|^{p'}\right)^{1/p'}\left(\sum_1^n k^{p-sp+1}\right)^{1/p}\\
&\le Mh^2\{o(n)\}^{1/p'}n^{1-s+2/p}\\
&= Mh^2n^{2-s+1/p}o(1).
\end{aligned}$$

If we choose $nh=1$ for our initial condition, the difference will be small provided

$$\text{(i)} \qquad n^{1+s-1/p}h \ge 1;$$
$$\text{(ii)} \qquad n^{2-s+1/p}h^2 \le M', \quad \text{say.}$$

For these conditions to be compatible we require $s \ge 1/p$. It follows that $|s_n(e^{ix})-\sum_1^\infty a_k e^{ikx}\frac{\sin kh}{kh}| \to 0$ as $h\to 0$, and the first statement is proved. The converse is proved in the same way. □

Remarks. (1) What the above really shows is that the case $s=1/p$ is the crucial value. If $s>1/p$ and $f\in B^s_{pp}$, then f is continuous, indeed f belongs to the little Lipschitz class λ_β, for every $\beta < s-1/p$, [**12**, p. 321]. Consequently, its Poisson integral is continuous on the closed disc. Furthermore we know from Theorem 10.8 of Chapter 2 of [**18**], that the Fourier series of f converges uniformly to f on the circle. Therefore the last two theorems are trivial in case $s>1/p$. However, if $s<1/p$, then B^s_{pp} is not contained in L^r for any $r > 1$ such that $1/r < 1/p - s$, [**12**, p. 321].

(2) In the proofs above we made essential use of Hölder's Inequality and the fact that $p > 1$. Further, we required $s \geq 1/p$. For the case $s = p = 1$, $f \in B^1_{11}$, the Fourier series of f is uniformly absolutely convergent on the circle. To see this, let us assume $f \in A^1_{11}$, as we may. We know that this implies that $f' \in H^1$ or $f(e^{ix})$ is absolutely continuous. But in that case $\sum_1^\infty |a_n| < \infty$ [**7**], and the uniform convergence follows. It is also immediate from the preceding and (10), that the symmetric derivative exists at every point on the circle.

5.2. An application.

Suppose that f is any summable function on T. Let us define the exceptional set $E(f)$ to be the set of points e^{ix} on T for which $\lim_{r\to 1} f(re^{ix})$, the radial limit of the Poisson integral of f on the unit disc D, fails to exist.

Our object here is to look at the size of the set $E(f)$ in case f belongs to some Besov space. This will require a notion of capacity associated with the space B^s_{pq}, denoted by $C(\cdot; B^s_{pq})$. We assume henceforth that $1 < p, q < \infty$. Consider the dual pairing

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(e^{ix})\overline{g(e^{ix})}\, dm.$$

It is well known that with this dual pairing, the dual space of $B = B^s_{pq}$ is $B^* = B^{-s}_{p'q'}$ which is a space of distributions and we refer to [**1**] and [**12**] for the definition. We state the following (dual) definition of capacity [**1**]. For a closed set $K \subset T$, the capacity of K is

$$C(K; B^s_{pq}) = \sup\big\{\mu(K) : \|\mu\|_{B^*} \leq 1\big\},$$

where the sup is taken over all positive measures μ on the circle which belong to the dual space $B^* = B^{-s}_{p'q'}$, and the norm is the norm in the dual space. The definition may now be extended to an arbitrary set G by means of a standard procedure; see Chapter 2 of [**1**]. If $p = q = 2$, $s = 1/2$, this capacity is equivalent to the logarithmic capacity. First we consider the exceptional set E' for F, $E' = \{e^{ix} : F(x) = \infty\}$.

Theorem 8. *Let $1 < p, q < \infty$, $0 < s < 1$ and $f \in A^s_{pq}$. With E' defined as above,*

$$C(E'; B^s_{pq}) = 0.$$

Proof: Letting $E'_m = \{e^{ix} : F(x) > m\}$, it is clear that $E' = \bigcap_m E'_m$. Since F is lower semi-continuous [**4**], it follows that E' is a G_δ set. Let μ be a positive measure in B^*. The dual pairing of F with μ, namely $\int_{-\pi}^{\pi} F\,d\mu$, is defined since $F \in B^s_{pq}$ by Theorem 4, and it satisfies

$$\int_{-\pi}^{\pi} F\,d\mu \leq ||F||_B ||\mu||_{B^*} < \infty.$$

But this implies that $\mu(E') = 0$ since $F(x) = \infty$ on E'. Since this holds for all μ, it follows that $C(E'; B^s_{pq}) = 0$ and the proof is complete. $\square$

The following corollary is significant only for $s \leq 1/p$ by an earlier remark.

Corollary 1. *Let $1 < p, q < \infty$, $0 < s \leq 1/p$ and $f \in B^s_{pq}$. Then*

$$C(E; B^s_{pq}) = 0.$$

Proof: It suffices to take $f \in A^s_{pq}$. Suppose that $\lim_{r\to 1} f(re^{ix})$ does not exist; it is readily verified that the radial variation of f at e^{ix} is infinite, $F(x) = \infty$. Consequently $E \subset E'$, and the result follows from the theorem. $\square$

Remark. For the special case in which $f \in B^s_{pp}$ with $s \geq 1/p$, $1 < p \leq 2$, we know from Theorems 6 and 7 that much more can be said: the exceptional set $E(f)$ coincides with the set of points e^{it} at which the Fourier series of f, $\sum_{-\infty}^{\infty} a_n e^{int}$, fails to converge, and also at which the symmetric derivative of f fails to exist.

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Department of Mathematics
NUI Maynooth
Maynooth, Co. Kildare
Ireland
E-mail address: David.Walsh@maths.nuim.ie

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